Problem Sheet 4 Additional Questions

9. *Alternative proof asymptotic result for* $\sum_{1\le n\le x} 2^{\omega(n)}$.

   Recall $2^\omega = 1 * Q_2$. In the notes we derived

$$\sum_{n\le x} 2^{\omega(n)} = \frac{1}{\zeta(2)} x\log x + O(x)\,,$$

   from the expression

$$\sum_{n\le x} 2^{\omega(n)} = \sum_{a\le x} Q_2(a) \sum_{b<x/a} 1 = \sum_{a\le x} Q_2(a)\left[\frac{x}{a}\right].$$

   Do this in an alternative manner, starting from

$$\sum_{1\le n\le x} 2^{\omega(n)} = \sum_{1\le a\le x}\ \sum_{1\le b<x/a} Q_2(b)\,.$$

10. *Alternative proof asymptotic result for* $\sum_{1\le n\le x} d(n^2)$.

    i) Prove that

$$\sum_{n\le x} \frac{\log(x/n)}{n} = \frac{1}{2}\log^2 x + O(\log x)\,. \qquad (18)$$

   Hint write $\log(x/n)$ as an integral and then interchange summation and integration.

   ii) Recall $d(n^2) = (1 * 2^\omega)(n)$. In the notes we proved Theorem 4.12.

$$\sum_{n\le x} d\big(n^2\big) = \frac{1}{2\zeta(2)} x\log^2 x + O(x\log x)\,,$$

   by starting from

$$\sum_{n\le x} d\big(n^2\big) = \sum_{a\le x} 2^{\omega(a)} \sum_{b\le x/a} 1.$$

   Do this in an alternative manner, starting from

$$\sum_{1\le n\le x} d\big(n^2\big) = \sum_{1\le a\le x}\ \sum_{1\le b<x/a} 2^{\omega(b)},$$

   and using Theorem 4.8.

11. *Alternative proof asymptotic result for* $\sum_{1\le n\le x} d^2(n)$.

Prove

$$\sum_{1\le n\le x} d^2(n) = \frac{1}{6\zeta(2)} x\log^3 x + O\big(x\log^2 x\big),$$

starting from the Convolution Method in the form

$$\sum_{1\le n\le x} d^2(n) = \sum_{1\le a\le x}\ \sum_{1\le b\le x/a} d\big(b^2\big)$$

Hint On Problem Sheet 2 you were asked to generalise (18) above, and prove

$$\sum_{1\le n\le x} \frac{\log^\ell (x/n)}{n} = \frac{1}{\ell+1}\log^{\ell+1} x + O\big(\log^\ell x\big), \qquad (19)$$

for any integer $\ell \ge 1$.

12. Define the arithmetic function k by $k(1) = 1$ and, for $n = p_1^{a_1}p_2^{a_2}...p_r^{a_r}$ a product of distinct primes, $k(n) = a_1a_2...a_r$, the product of the exponents.

Prove that

$$\sum_{n=1}^{\infty} \frac{k(n)}{n^s} = \frac{\zeta(s)\,\zeta(2s)\,\zeta(3s)}{\zeta(6s)}$$

for $\operatorname{Re} s > 1$.

b. Let q_2 be the characteristic function of square-full numbers, so $q_2(1) = 1$ and, for $n = p_1^{a_1}p_2^{a_2}...p_r^{a_r}$ a product of distinct primes, $q_2(n) = 1$ if all $a_i \ge 2$, and $q_2(n) = 0$ if some $a_i = 1$.

Prove that $k = 1 * q_2$.

c. Prove that

$$\sum_{n\le x} \frac{q_2(n)}{n} = \zeta(2)\,\frac{\zeta(3)}{\zeta(6)} + O\left(\frac{1}{\sqrt{x}}\right).$$

Hint Do not use Partial summation on the previous question but start from

$$\sum_{n\le x} \frac{q_2(n)}{n} = \sum_{d\le x} \frac{h(d)}{d} \sum_{a\le x/d} \frac{sq(a)}{a}.$$

d. Prove that
$$\sum_{n\le x} k(n) = \frac{\zeta(2)\,\zeta(3)}{\zeta(6)} x + O\big(\sqrt{x}\big)\,.$$